The Lawson homology and Deligne-Beilinson cohomology for Fulton-MacPherson configuration spaces

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Abstract

In this paper, we compute the Lawson homology groups and Deligne-Beilinson cohomology groups for Fulton-MacPherson configuration spaces. The explicit formulas are given.

Contents

1 Introduction

In this paper, all varieties are defined over $\mathbb{C}$. Let X be an d-dimensional projective variety. Let $\mathcal{Z}_p(X)$ be the space of algebraic p-cycles on X.

The **Chow group** $\mathrm{Ch}_p(X)$ of p-cycles is defined by $\mathcal{Z}_p(X)$ modulo the rational equivalence. For general background on Chow groups, the reader is referred to Fulton's book [Fu].

The **Lawson homology** $L_pH_k(X)$ of p-cycles is defined by

$$L_pH_k(X) := \pi_{k-2p}(\mathcal{Z}_p(X)) \quad \text{for} \quad k \geq 2p \geq 0,$$

where $\mathcal{Z}_p(X)$ is provided with a natural topology (cf. [F], [L1]). For general background on Lawson homology, the reader is referred to [L2].

It is convenient to extend the definition of Lawson homology by setting

$$L_pH_k(X) = L_0H_k(X), \quad \text{if} \quad p < 0.$$

It was proved in [H] that, for any smooth projective variety X, the formula on Lawson homology for a blowup holds:

Theorem 1.1 ([H]) *Let X be smooth projective variety and $Y \subset X$ be a smooth subvariety of codimension $r \geq 2$. Let $\sigma : \tilde{X}_Y \to X$ be the blowup of X along Y, $\pi : D := \sigma^{-1}(Y) \to Y$ the natural map, and $i : D \to \tilde{X}_Y$ the exceptional divisor of the blowup. Then for integers p, k with $k \geq 2p \geq 0$, there is an isomorphism*

$$I_{p,k} : \left\{ \bigoplus\nolimits_{1\leq j\leq r-1} L_{p-j}H_{k-2j}(Y) \right\} \oplus L_pH_k(X) \cong L_pH_k(\tilde{X}_Y).$$

Now we give minimal notations for the Fulton-MacPherson configuration spaces enough for stating the main theorem (see section 2.2 for a construction of the Fulton-MacPherson configuration spaces by a sequence of blowups.)

Let X be a smooth projective variety of dimension d and let $n \geq 1$ be an integer. Consider the cartesian product $X^n := X \times \cdots \times X$ of n copies of X. Denote by Δ_I the diagonal in X^n where $x_i = x_j$ if $i, j \in I$.

The *configuration space* $F(X, n)$ is the complement of all diagonals in X^n, i.e.,

$$F(X, n) = X^n \setminus \cup_{|I|\geq 2}\Delta_I = \{(x_1, \ldots, x_n) \in X^n : x_i \neq x_j, \forall i \neq j\}.$$

For each subset $I \in [n] := \{1, \ldots, n\}$ with at least two elements, denote by $\mathrm{Bl}_\Delta(X^I)$ the blowup of the corresponding cartesian product X^I along its small diagonal. In [FuM], Fulton and MacPherson have given the definition of their compactification $X[n]$ as follows.

Theorem 1.2 (Fulton-MacPherson) *The closure of the natural locally closed embedding*

$$i : F(X, n) \hookrightarrow X^n \times \prod_{|I|\geq 2} Bl_\Delta(X^I)$$

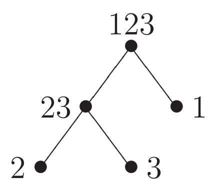
is smooth, and the boundary is a simple normal crossing divisor. The closure is called the Fulton-MacPherson configuration space, denoted by $X[n]$.

We call two subsets $I, J \subseteq [n] := \{1, 2, \ldots, n\}$ are *overlapped* if $I \cap J$ is a nonempty proper subset of I and of J.

A *nest* $\mathcal{S}$ is a set of subsets of $[n]$ such that any two elements $I \neq J \in \mathcal{S}$ are not overlapped, and all singletons $\{1\}, \ldots, \{n\}$ are in $\mathcal{S}$. Notice that the nest defined here, unlike the one defined in [FuM], contains singletons.

Given a nest $\mathcal{S}$, define $\mathcal{S}^\circ = \mathcal{S} \setminus \{\{1\}, \ldots, \{n\}\}$. In the description of nests by forests below, $\mathcal{S}^\circ$ corresponds to the forest $\mathcal{S}$ cutting of all leaves.

A nest $\mathcal{S}$ naturally corresponds to a not necessarily connected tree (which is also called a *forest* or a *grove*), each node of which is labeled by an element in $\mathcal{S}$. For example, the following forest corresponds to a nest $\mathcal{S} = \{1, 2, 3, 23, 123\}$.



Denote by $c(\mathcal{S})$ the number of connected components of the forest, i.e., the number of maximal elements of $\mathcal{S}$. Denote by $c_I(\mathcal{S})$ (or c_I if no ambiguity arise) the number of maximal elements of the set $\{J \in \mathcal{S} | J \subsetneq I\}$, i.e. the number of sons of the node I. In the above example, $c(\mathcal{S}) = 1$, $c_{123} = c_{23} = 2$.

For a nest $\mathcal{S} \neq \{\{1\}, \ldots, \{n\}\}$ (i.e. $\mathcal{S}^\circ \neq \emptyset$), define a set $M_{\mathcal{S}}$ of lattice points in the integer lattice $\mathbb{Z}^{\mathcal{S}^\circ}$ as follows

$$M_{\mathcal{S}} := \{\underline{\mu} = \{\mu_I\}_{I \in \mathcal{S}^\circ} : 1 \leq \mu_I \leq d(c_I - 1) - 1\}.$$

(Recall that $d = \dim X$, $c_I = c_I(\mathcal{S})$) and define $||\underline{\mu}|| := \sum_{I \in \mathcal{S}^\circ} \mu_I$, $\forall \underline{\mu} \in M_{\mathcal{S}}$.

For $\mathcal{S} = \{\{1\}, \ldots, \{n\}\}$, assume $M_{\mathcal{S}} = \{\underline{\mu}\}$ with $\|\underline{\mu}\| = 0$.

It was proved in [Li] that, for any smooth projective variety X the following holds:

Theorem 1.3 ([Li]) *Let X be a smooth projective variety defined over $\mathbb{C}$. Then for each $p \geq 0$, there is an isomorphism of Chow groups:*

$$\mathrm{Ch}_p(X[n]) \cong \bigoplus_{\mathcal{S}} \bigoplus_{\underline{\mu} \in M_{\mathcal{S}}} Ch_{p-||\underline{\mu}||}(X^{c(\mathcal{S})}).$$

where $\mathcal{S}$ runs through all nests of $[n]$.

The first main result in this paper is the following

Theorem 1.4 *Let X be a smooth projective variety defined over $\mathbb{C}$. Then for each pair of integers p, k, $k \geq 2p \geq 0$, there is an isomorphism of Lawson homology groups:*

$$L_pH_k(X[n]) \cong \bigoplus_{\mathcal{S}} \bigoplus_{\underline{\mu} \in M_{\mathcal{S}}} L_{p-||\underline{\mu}||}H_{k-2||\underline{\mu}||}(X^{c(\mathcal{S})}).$$

where $\mathcal{S}$ runs through all nests of $[n]$.

Remark 1.1 *When $p = 0$, Theorem 1.4 reduces to the formula of singular homology groups with integer coefficient for $X[n]$. In particular, the integer singular homology of $X[n]$ depends only the integer singular homology of X.*

As a corollary, we have the following more explicit formula:

Corollary 1.1 *Let X be a smooth projective variety defined over $\mathbb{C}$. Then for each pair of integers p, k, $k \geq 2p \geq 0$, there is an isomorphism of Lawson homology groups:*

$$L_pH_k(X[n]) \cong \bigoplus_{\substack{1\leq m\leq n \\ 0\leq i\leq p}} L_{p-i}H_{k-2i}(X^m)^{\oplus[\frac{x^it^n}{n!}]\frac{N^m}{m!}}.$$

where N and $\oplus[\frac{x^it^n}{n!}]\frac{N^m}{m!}$ are defined in (1).

Let X be a complex manifold of complex dimension d. Let Ω_X^k the sheaf of holomorphic k-form on X. The **Deligne complex of level p** is the complex of sheaves

$$\underline{\mathbb{Z}}_{\mathcal{D}}(p) : 0 \to \mathbb{Z} \stackrel{(2i\pi)^p}{\to} \Omega_X^0 \to \Omega_X^1 \to \Omega_X^2 \to \cdots \to \Omega_X^{p-1} \to 0$$

The **Deligne-Beilinson cohomology** of X in level p we mean the hypercohomology of this complex:

$$H_{\mathcal{D}}^*(X, \mathbb{Z}(p)) := \mathbb{H}^*(X, \underline{\mathbb{Z}}_{\mathcal{D}}(p))$$

For Deligne-Beilinson cohomology $H_{\mathcal{D}}^k(-, \mathbb{Z}(p))$, we obtain the following result:

Theorem 1.5 *Let X be a smooth projective variety defined over $\mathbb{C}$. Then for each pair of integers p, k, there is an isomorphism of Deligne-Beilinson cohomology groups:*

$$H_{\mathcal{D}}^k(X[n], \mathbb{Z}(p)) \cong \bigoplus_{\mathcal{S}} \bigoplus_{\underline{\mu}\in M_{\mathcal{S}}} H_{\mathcal{D}}^{k-2||\underline{\mu}||}(X^{c(\mathcal{S})}, \mathbb{Z}(p - ||\underline{\mu}||)).$$

The main tools used to prove the main result are: The formula on the Lawson homology for a blowup proved in [H] and the method in computing the Chow groups of the Fulton-MacPherson configuration space in [Li].

2 Some fundamental materials

2.1 Lawson homology

Recall that for a morphism $f : U \to V$ between projective varieties, there exist induced homomorphism $f_* : L_pH_k(U) \to L_pH_k(V)$ for all $k \geq 2p \geq 0$. Furthermore, it has been

shown by C. Peters [Pe] that if U and V are smooth and projective, there are Gysin "wrong way" homomorphism $f^*: L_pH_k(V) \to L_{p-c}H_{k-2c}(U)$, where $c = \dim(V) - \dim(U)$.

Let X be a smooth projective variety and $i_0 : Y \hookrightarrow X$ a smooth subvariety of codimension r. Let $\sigma : \tilde{X}_Y \to X$ be the blowup of X along Y, $\pi : D = \sigma^{-1}(Y) \to Y$ the natural map, and $i : D = \sigma^{-1}(Y) \hookrightarrow \tilde{X}_Y$ the exceptional divisor of the blowing up. Set $U \equiv X - Y \cong \tilde{X}_Y - D$. Denote by j_0 the inclusion $U \subset X$ and j the inclusion $U \subset \tilde{X}_Y$. Note that $\pi : D = \sigma^{-1}(Y) \to Y$ makes D into a projective bundle of rank $r-1$, given precisely by $D = \mathrm{P}(N_{Y/X})$ and we have (cf. [V2], pg.271)

$$\mathcal{O}_{\tilde{X}_Y}(D)|_D = \mathcal{O}_{\mathrm{P}(N_{Y/X})}(-1).$$

Denote by h the class of $\mathcal{O}_{\mathrm{P}(N_{Y/X})}(-1)$ in $\mathrm{Pic}(D)$. We have $h = -D_{|D}$ and $-h = i^*i_* : L_qH_m(D) \to L_{q-1}H_{m-2}(D)$ for $0 \leq 2q \leq m$ ([[FG], Theorem 2.4], [[Pe], Lemma 11]). The last equality can be equivalently regarded as a Lefschetz operator

$$-h = i^*i_* : L_qH_m(D) \to L_{q-1}H_{m-2}(D), \quad 0 \leq 2q \leq m.$$

The proof of the main result are based on the following Theorems:

Theorem 2.1 (Lawson homology for a blowup) *Let X be smooth projective manifold and $Y \subset X$ be a smooth subvariety of codimension r. Let $\sigma : \tilde{X}_Y \to X$ be the blowup of X along Y, $\pi : D = \sigma^{-1}(Y) \to Y$ the natural map, and $i : D = \sigma^{-1}(Y) \to \tilde{X}_Y$ the exceptional divisor of the blowing up. Then for each p, k with $k \geq 2p \geq 0$, we have the following isomorphism*

$$I_{p,k} : \left\{ \bigoplus_{1 \leq j \leq r-1} L_{p-j}H_{k-2j}(Y) \right\} \oplus L_pH_k(X) \xrightarrow{\cong} L_pH_k(\tilde{X}_Y)$$

given by

$$I_{p,k}(u_1, \cdots, u_{r-1}, u) = \sum_{j=1}^{r-1} i_*(h^j \cdot \pi^* u_j) + \sigma^* u.$$

2.2 The Fulton-MacPherson configuration spaces

Fulton and MacPherson have constructed in [FuM] a compactification of the configuration space of n distinct labeled points in a non-singular algebraic variety X. It is related to several areas of mathematics. In their original paper, Fulton and MacPherson use it to construct a differential graded algebra which is a model for $F(X, n)$ in the sense of Sullivan [FuM]. Axelrod-Singer constructed the compactification in the setting of smooth manifolds. $\mathbb{P}^1[n]$ is related to the Deligne-Mumford compactification $\overline{M}_{0,n}$ of the moduli space of nonsingular genus-0 projective curves.

Now we explain an explicit inductive construction of this compactification given in [FuM]. $X[2]$ is the blowup of X^2 along the diagonal Δ_{12}. $X[3]$ is a sequence of blowups of $X[2] \times X$ along non-singular subvarieties corresponding to $\{\Delta_{123}; \Delta_{13}, \Delta_{23}\}$. More specifically, denote by π the blowup $X[2] \times X \to X^3$, we blow up first along $\pi^{-1}(\Delta_{123})$, then along the strict transforms of Δ_{13} and Δ_{23} (the two strict transforms are disjoint, so they can be blown up in any order). In general, $X[n+1]$ is a sequence of blowups of $X[n] \times X$ along smooth subvarieties corresponding to all diagonals Δ_I where $|I| \geq 2$ and $(n+1) \in I$.

Later, a symmetric construction of $X[n]$ has been given by several people: De Concini and Procesi [DP], MacPherson and Procesi [MP], and Thurston [Th]. To construct $X[n]$ we can blow up along diagonals by the order of ascending dimension, which is different from the non-symmetric order of the original construction. For example, $X[4]$ is the blowup of X^4 along diagonals corresponding to:

$$1234; 123, 124, 134, 234; 12, 13, \ldots, 34.$$

Compare it with the order in [FuM]:

$$12; 123; 13, 23; 1234; 124, 134, 234; 14, 24, 34.$$

It is proved in [Li] that, for any smooth projective variety X the following holds:

Theorem 2.2 ([Li]) *Let X be a smooth projective variety defined over $\mathbb{C}$. Then for each $p \geq 0$, there is an isomorphism of Chow groups:*

$$\mathrm{Ch}^p(X[n]) \cong \bigoplus_{\mathcal{S}} \bigoplus_{\underline{\mu} \in M_{\mathcal{S}}} \mathrm{Ch}^{p - ||\underline{\mu}||}(X^{c(\mathcal{S})}).$$

where $\mathcal{S}$ runs through all nests of $[n]$.

Notice that we use upper indices for the Chow groups in the above theorem. By changing variable μ_I to $d(c_I - 1) - \mu_I$, we get exactly Theorem 1.3 appeared in the introduction.

Remark 2.1 *The above theorem proved in [Li] holds for non-singular projective varieties X over any algebraic closed field.*

Equivalently, but more explicitly, the Chow groups $X[n]$ of can be calculated by using exponential generating functions. Here we adopt R. Stanley's notation $[x^i]F(x)$ as the coefficient of x^i in the power series $F(x)$, which is generalized in an obvious way to the following situation [St]:

$$[\frac{x^i t^n}{n!}] \sum_{j,q} a_{jq} \frac{x^j t^q}{q!} = a_{in}. \tag{1}$$

Corollary 2.1 *If $h_i(x)$ are polynomials whose exponential generating function $N(x,t) = \sum_{i\geq 1} h_i(x)\frac{t^i}{i!}$ satisfies the identity*

$$(1-x)x^d t + (1-x^{d+1}) = \exp\,(x^d N) - x^{d+1}\exp\,(N),$$

then we have

$$\mathrm{Ch}_p(X[n]) = \bigoplus_{\substack{1\leq m\leq n\\ 0\leq i\leq p}} \mathrm{Ch}_{p-i}(X^m)^{\oplus[\frac{x^i t^n}{n!}]\frac{N^m}{m!}}.$$

□

2.3 Deligne-Beilinson cohomology

Let X be a complex manifold of complex dimension d. Let Ω_X^k the sheaf of holomorphic k-form on X. The **Deligne complex of level p** is the complex of sheaves

$$\underline{\mathbb{Z}}_{\mathcal{D}}(p) : 0 \to \mathbb{Z} \stackrel{(2i\pi)^p}{\to} \Omega_X^0 \to \Omega_X^1 \to \Omega_X^2 \to \cdots \to \Omega_X^{p-1} \to 0$$

The **Deligne-Beilinson cohomology** of X in level p we mean the hypercohomology of this complex:

$$H_{\mathcal{D}}^*(X,\mathbb{Z}(p)) := \mathbb{H}^*(X,\underline{\mathbb{Z}}_{\mathcal{D}}(p))$$

There is a multiplication of complexes

$$\nu : \mathbb{Z}(p)_{\mathcal{D}} \otimes \mathbb{Z}(q)_{\mathcal{D}} \to \mathbb{Z}(p+q)_{\mathcal{D}}$$

defined as follows

$$\nu(x \bullet y) = \begin{cases} x\cdot y, & \text{if} \quad \deg\, x = 0 \\ x\wedge dy, & \text{if} \quad \deg\, x > 0 \quad \textit{and} \quad \deg\, y = q > 0 \\ 0, & \text{otherwise} \end{cases}$$

This gives a product structure on the Deligne–Beilinson cohomology as follows

$$\cup : H_{\mathcal{D}}^k(X,\mathbb{Z}(p)) \otimes_{\mathbb{Z}} H_{\mathcal{D}}^{k'}(X,\mathbb{Z}(q)) \to H_{\mathcal{D}}^{k+k'}(X,\mathbb{Z}(p+q)). \tag{2}$$

For details, the reader is referred to [EV].

Let X be an d-dimensional compact Kähler manifold. The Hodge filtration

$$\cdots \subseteq F^p H^k(X,\mathbb{C}) \subseteq F^{p-1}H^k(X,\mathbb{C}) \subseteq \cdots \subseteq F^0 H^k(X,\mathbb{C}) = H^k(X,\mathbb{C})$$

is defined by

$$F^pH^k(X,\mathbb{C}) = \oplus_{i\geq p}H^{i,k-i}(X).$$

We denote by p_X^k the natural quotient map $p_X^k : H^k(X,\mathbb{C}) \to H^k(X,\mathbb{C})/F^pH^k(X,\mathbb{C})$.

It was proved (cf. [EV], Corollary 2.4; [V1], Proposition 12.26) that

$$\cdots \to H^{k-1}(X,\mathbb{C})/F^pH^{k-1}(X,\mathbb{C}) \to H^k_{\mathcal{D}}(X,\mathbb{Z}(p)) \to H^k(X,\mathbb{Z}) \to H^k(X,\mathbb{C})/F^pH^k(X,\mathbb{C}) \to \cdots \quad (3)$$

Now let X be an d-dimensional projective variety over $\mathbb{C}$ and $i_0 : Y \hookrightarrow X$ a smooth subvariety of codimension $r \geq 2$. Let $\sigma : \tilde{X}_Y \to X$ be the blowup of X along Y, $\pi : D = \sigma^{-1}(Y) \to Y$ the natural map, and $i : D = \sigma^{-1}(Y) \hookrightarrow \tilde{X}_Y$ the exceptional divisor of the blowup. Set $U := X - Y \cong \tilde{X}_Y - D$. Denote by j_0 the inclusion $U \subset X$ and j the inclusion $U \subset \tilde{X}_Y$. Note that $\pi : D = \sigma^{-1}(Y) \to Y$ makes D into a projective bundle of rank $r-1$, given precisely by $D = \mathrm{P}(N_{Y/X})$ and we have (cf. [[V2], pg. 271])

$$\mathcal{O}_{\tilde{X}_Y}(D)|_D = \mathcal{O}_{\mathrm{P}(N_{Y/X})}(-1).$$

Denote by h the class of $\mathcal{O}_{\mathrm{P}(N_{Y/X})}(-1)$ under the first Chern class $c_1 : H^1(D,\mathcal{O}_D^*) \to H^2_{\mathcal{D}}(D,\mathbb{Z}(1))$ (cf. [[EV], p. 88]).

The following proposition was proved in [EV].

Proposition 2.1 ([EV], Prop. 8.5) *The Deligne-Beilinson cohomology $H^k_{\mathcal{D}}(D,\mathbb{Z}(p))$ of the projective bundle $\pi : D \to Y$ is given by the following isomorphism:*

$$\bigoplus_{0\leq j\leq r-1} H^{k-2j}_{\mathcal{D}}(Y,\mathbb{Z}(p-j)) \xrightarrow{\cong} H^k_{\mathcal{D}}(D,\mathbb{Z}(p))$$

Remark 2.2 *We omit the cup product of elements in $H^{k-2j}_{\mathcal{D}}(Y,\mathbb{Z}(p-j))$ with h^j.*

Moreover, Barbieri-Viale proved the following blowup formula for Deligne-Beilinson cohomology:

Theorem 2.3 ([Bv])) *Let $X, Y, D, \tilde{X}_Y, Y$ be as above. Then for each p, k with $p \geq r \geq 0$, we have the following isomorphism*

$$I_{p,k} : \left\{ \bigoplus_{1\leq j\leq r-1} H^{k-2j}_{\mathcal{D}}(Y,\mathbb{Z}(p-j)) \right\} \oplus H^k_{\mathcal{D}}(X,\mathbb{Z}(p)) \xrightarrow{\cong} H^k_{\mathcal{D}}(\tilde{X}_Y,\mathbb{Z}(p)). \quad (4)$$

Remark 2.3 *Barbieri-Viale proved a general result, including the blowup formula for étale cohomology, to Theorem 2.3.*

3 Lawson homology for Fulton-MacPherson configuration spaces

In this section, we give a proof of Theorem 1.4. According to the construction, the Fulton-MacPherson configuration space $X[n]$ is obtained by a sequence of blowups along all diagonals Δ_I in a suitable order. Each of them is a blowup of a *smooth* projective variety along a *smooth* projective subvariety. Therefore, we can calculate the Lawson homology groups of $X[n]$ by successively applying the blowup formula for Lawson homology (Theorem 2.1).

We have the following

Theorem 3.1 *Let X be a smooth projective variety defined over $\mathbb{C}$. Then for each pair of integers p, k, $k \geq 2p \geq 0$, there is an isomorphism of Lawson homology groups:*

$$L_pH_k(X[n]) \cong \bigoplus_{\mathcal{S}} \bigoplus_{\underline{\mu} \in M_{\mathcal{S}}} L_{p-||\underline{\mu}||}H_{k-2||\underline{\mu}||}(X^{c(\mathcal{S})}).$$

Proof. This follows essentially from the construction of the Fulton-MacPherson configuration space $X[n]$ and the blowup formula for Lawson homology groups. The detailed computation for explicit formulas will be given in the corollary below.

□

More explicitly, we have the following

Corollary 3.1 *Let X be a smooth projective variety defined over $\mathbb{C}$. Then for each pair of integers p, k, $k \geq 2p \geq 0$, there is an isomorphism of Lawson homology groups:*

$$L_pH_k(X[n]) \cong \bigoplus_{\substack{1 \leq m \leq n \\ 0 \leq i \leq p}} L_{p-i}H_{k-2i}(X^m)^{\oplus[\frac{x^it^n}{n!}]\frac{N^m}{m!}}.$$

where N and $\oplus[\frac{x^it^n}{n!}]\frac{N^m}{m!}$ are the same as those in Corollary 2.1.

Proof. Let $h_n(x)$ be the polynomial

$$\sum_{\substack{\mathcal{S},\ \underline{\mu} \\ c(\mathcal{S})=1}} x^{||\underline{\mu}||}.$$

Given a fixed nest $\mathcal{S}$ with n leaves and $c(\mathcal{S}) = 1$, its contribution to $h_n(x)$ is the product of σ_{c_I-1}, where I goes through all non-leaves of $\mathcal{S}$ (if $\mathcal{S}$ has no non-leaves, i.e., it contains only singletons, then the contribution is 1). Therefore we have the following recurrence formula

$$h_n(x) = \sum_{\{I_1,\ldots,I_k\} \text{ partition of } [n]} h_{|I_1|}h_{|I_2|}\ldots h_{|I_k|}\sigma_{k-1}.$$

where $\sigma_k = \sum_{i=1}^{dk-1} x^i$ for $k > 0$, and $\sigma_0 = 0$.

By the Compositional Formula of exponential generating functions (cf. [St], Theorem 5.1.4), the generating function $N(t) := \sum_{i\geq 1} h_i \frac{t^i}{i!}$ of h_n satisfies the identity

$$N - t + 1 = E_g(N),$$

where $E_g(N) = 1 + \sum_{i>0} \sigma_{i-1} N^i$.

Since $\sigma_j = (x^{jd} - x)/(x-1)$, calculation shows

$$E_g(N) = 1 + N + \frac{1}{x-1}\big[\frac{1}{x^d}(e^{x^d N - 1} - 1) - xe^N + x\big].$$

Put it in the above identity, we have

$$(1-x)x^d t + (1 - x^{d+1}) = \exp\,(x^d N) - x^{d+1}\exp\,(N).$$

For a partition $\Pi = \{I_1, ..., I_k\}$ of $[n]$, the number of times of $h(\Delta_\Pi)(i)$ appear in the decomposition of $h(X[n])$ is equal to $[x^k](h_{|I_1|}(x)...h_{|I_k|}(x))$. Add up this number for all partitions with k blocks, we will get the number of times of $h(X^k)(i)$ appear in the decomposition, denoted by $a_{k,i}$.

Denote

$$F_n(y) = \sum_{\{I_1,...,I_k\} \text{ partition of } [n]} h_{|I_1} h_{|I_2} ... h_{|I_k} y^k.$$

Then the coefficient $[y^k]F_n(y) = \sum a_{k,i}x^i$. Use the Compositional Formula again,

$$F_n = [\frac{t^n}{n!}]\exp\,(yN).$$

Therefore

$$\begin{aligned} [y^k]F_n(y) &= [y^k][\tfrac{t^n}{n!}]\exp\,(yN) \\ &= [\tfrac{t^n}{n!}][y^k]\exp\,(yN) \\ &= [\tfrac{t^n}{n!}]\tfrac{N^k}{k!}. \end{aligned}$$

Now the results follow from Theorem 3.1.

□

Similarly, we compute the Deligne-Beilinson cohomology for Fulton-MacPherson configuration spaces.

Theorem 3.2 *Let X be a smooth projective variety defined over $\mathbb{C}$. Then for each pair of integers p, k, there is an isomorphism of Deligne-Beilinson cohomology groups:*

$$H^k_{\mathcal{D}}(X[n], \mathbb{Z}(p)) \cong \bigoplus_{\mathcal{S}} \bigoplus_{\underline{\mu}\in M_{\mathcal{S}}} H_{\mathcal{D}}^{k-2||\underline{\mu}||}\big(X^{c(\mathcal{S})}, \mathbb{Z}(p - ||\underline{\mu}||)\big).$$

Proof. The method of the proof is the same as that in Theorem 3.1. We get the result by using the explicit construction of Fulton-MacPherson configuration spaces and Theorem 2.3. □

Remark 3.1 *By using the same method, we can compute the étale cohomology for Fulton-MacPherson configuration spaces.*

Remark 3.2 *The decomposition of Lawson homology (Theorem 3.1) and Deligne - Beilinson cohomology (Theorem 3.2) of the Fulton-MacPherson configuration spaces can be generalized without any difficulty to the wonderful compactifications of arrangements of subvarieties, since the latter compactifications can also be constructed by a sequence of blowups along smooth centers (for definition and construction of these compactifications, see [Li]).*

4 Examples

1. The Lawson homology group of $X[2]$.

   The morphism $\pi : X[2] \to X^2$ is a blowup along the diagonal Δ_{12}. Corollary 3.1 asserts

   $$L_pH_k(X[2]) \cong L_pH_k(X^2) \oplus \bigoplus_{j=1}^{d-1} L_{p-j}H_{k-2j}(X).$$

2. The Lawson homology group of $X[3]$.

   Note that $X[3]$ is the blowup of X^3 first along small diagonal Δ_{123}, then along three disjoint proper transforms of diagonals Δ_{12}, Δ_{13} and Δ_{23}.

   Apply again Corollary 3.1, we have

   $$L_pH_k(X[3]) \cong L_pH_k(X^3) \oplus \bigoplus_{j=1}^{d-1} \left(L_{p-j}H_{k-2j}(X^2)\right)^{\oplus 3}$$

   $$\oplus \bigoplus_{j=1}^{2d-1} \left(L_{p-j}H_{k-2j}(X)\right)^{\oplus min\{3i-2, 6d-3i-2\}}$$

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